

COORDINATES, RETRACTS AND AUTOMORPHISMS

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ABSTRACT. Let K be a field of characteristic zero, $K[x, y]$ be the polynomial ring in two variables. Let $\phi = (f, g)$ be an endomorphism of $K[x, y]$. It is proved that if ϕ maps each coordinate to a generator of some proper retract, then it is an automorphism. As a corollary, the retract preserving problem is solved for both polynomial ring over K and free algebra over an arbitrary field when $n = 2$.

1. INTRODUCTION

Let K be a field of characteristic zero, $P_n = K[x_1, \dots, x_n]$ be the polynomial ring in n variables and σ be an endomorphism. By σ "preserving" things of type A , we mean that σ maps things of type A to things of the same type. For some preserving problems of polynomial and free algebras, see [3, 8, 5, 4, 6]. Recall that a subring R of P_n is a retract if there exists some idempotent endomorphism π (π is idempotent means $\pi^2 = \pi$) such that $\pi(P_n) = R$. The endomorphism is called the corresponding retraction of R . For more information, please refer to [3, 2, 7]. Then the corresponding "retract preserving problem" is raised naturally:

Problem 1.1 (Jie-Tai, Yu). *Is an endomorphism that maps each proper retract to a proper retract an automorphism ?*

For free algebras, the parallel problem is not true with the following simple counterexample.

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Example 1.2. Let F be an arbitrary field, $F\langle x, y \rangle$ be the free algebra in two variables and $\phi = (x, y + xy - yx)$ be an endomorphism. Then it is not an automorphism obviously since the leading terms are not algebraically dependent. According to [2, 7], if R is a proper retract of $F\langle x, y \rangle$, then $R = F[r]$ for some $r \in F\langle x, y \rangle$. Let $\pi = (s(r), t(r))$ be the corresponding retraction, and then

$$\pi(r) = r(s(r), t(r)) = r.$$

Define $\phi(r) = r'$, and then $r'(x, y) = r(x, y) + c(x, y)$ where $c(x, y)$ is a commutator. Since $r(s(r), t(r)) = r$, $r(s(r'), t(r')) = r'$. Hence if we define $\pi' = (s(r'), t(r'))$, then $\pi'(r') = r(s(r'), t(r')) + c(s(r'), t(r'))$. However, since $s(r')$ and $t(r')$ are algebraically dependent, $c(s(r'), t(r')) = 0$, and hence $\pi'(r') = r'$ which implies that $F[r']$ is a retract with the corresponding retraction π' .

In this paper, a positive solution is given to Problem 1.1 when $n = 2$. Following is the main theorem .

Theorem 1.3 (Main). *Let K be a field of characteristic zero, $K[x, y]$ be the polynomial ring in two variables and ϕ be an endomorphism that maps each coordinate to a generator of some proper retract. Then ϕ is an automorphism.*

Corollary 1.4. *Let K be a field of characteristic zero. Then an endomorphism ϕ which preserves the proper retracts of P_2 is an automorphism.*

Proof. Since each coordinate is also a generator of some proper retract, ϕ maps each coordinate to a generator of some proper retract, and hence it is an automorphism by Theorem 1.3.

□

2. PROOF OF THE MAIN THEOREM

Throughout this section, we always assume that K is a field of characteristic zero. Let $\phi = (f, g)$ be an endomorphism which maps each coordinate to a generator of some proper retract.

Lemma 2.1. *Let p be a coordinate and $p' = \phi(p)$. Let $\pi = (s(p'), t(p'))$ be some corresponding retraction of $K[p']$ and $\pi' = (s(z), t(z))$ where $s(z), t(z) \in K[z]$. Then $K[\pi'(f), \pi'(g)] = K[z]$.*

Proof. Since π is the retraction, then $\pi(p') = p'$, or $p'(s(p'), t(p')) = p'$. Since

$$p'(s(p'), t(p')) = p'(s(z), t(z)) \mid_{z=p'},$$

$\pi'(p')$ has to be equal to z since $p' \notin K$. Hence $\pi' \circ \phi(p) = z$ which implies $K[\pi'(f), \pi'(g)] = K[z]$. □

By [7], since $\phi(x) = f$ is a generator of some proper retract, then there exists some automorphism σ such that $\sigma(f) = x + y \cdot h_1(x, y)$. Let $\sigma(g) = y \cdot h_2(x, y) + h(x)$ and $\sigma' = (x, y - h(x))$, then $\sigma \circ \phi \circ \sigma' = (x + y \cdot h_1(x, y), y \cdot h_2(x, y))$. Obviously that $\sigma \circ \phi \circ \sigma'$ also maps each coordinate to a generator of a proper retract and ϕ is an automorphism if and only if $\sigma \circ \phi \circ \sigma'$ is an automorphism. Hence we assume that $\phi = (x + y \cdot h_1(x, y), y \cdot h_2(x, y))$ for some $h_1, h_2 \in K[x, y]$ and $h_2 \neq 0$.

Lemma 2.2. *To any n , there exists some $\pi_n = (s_n(z), t_n(z))$ where $\deg(s_n) \cdot \deg(t_n) > 0$ and $\deg(s_n) > n \cdot \deg(t_n)$ such that*

$$K[\pi_n(f), \pi_n(g)] = K[z].$$

Proof. Assume not, and then there exists some positive integer N such that if $\deg(s(z)) \cdot \deg(t(z)) > 0$ and $K[f(s, t), g(s, t)] = K[z]$, then $\deg(s) \leq N \cdot \deg(t)$.

Now consider the coordinate $y + (x + y^M)^2$ where $M > \max\{\deg(h_1) + 2, N, 1 + (N + 1)\deg(h_1)\}$. By Lemma 2.1 there exists some $\pi = (s(z), t(z))$ such that $\pi \circ \phi(y + (x + y^M)^2) = z$, or

$$\pi(yh_2 + (x + yh_1 + y^M h_2^M)^2) = z.$$

If $t(z)$ is a constant, then $\pi(h_2)$ can not be a constant since if so, then $\pi(yh_2)$ is a constant, and hence $z - \pi(yh_2) = (\pi(x + yh_1 + y^M h_2^M))^2$ which is impossible. Hence $\deg(\pi(yh_2)) \geq 1$ and $\deg(\pi(y^M h_2^M)) \geq M$. Since $K[s(z), t(z)] = K[z]$, then $s(z) = az + b$ where $a, b \in K$ and $a \neq 0$ ($t(z)$ is a constant). Hence $\deg(\pi(yh_1)) \leq \deg_x(h_1) < M$, and then

$$\deg(\pi(yh_2 + (x + yh_1 + y^M h_2^M)^2)) = \deg(\pi(y^{2M} h_2^{2M})) > 2,$$

which contradicts to $\pi(yh_2 + (x + yh_1 + y^M h_2^M)^2) = z$. Hence $t(z) \notin K$.

If $s(z) \in K$, then $t(z) = az + b$ where $a, b \in K$ and $a \neq 0$. If $\pi(h_2) = 0$, then

$$\pi(yh_2 + (x + yh_1 + y^M h_2^M)^2) = (\pi(x + yh_1))^2 \neq z.$$

Hence $\pi(h_2) \neq 0$, and then $\deg(\pi(yh_2)) \geq 1$. Since $\deg(\pi(yh_1)) \leq 1 + \deg_y(h_1) < M$,

$$\deg(\pi(x + yh_1 + y^M h_2^M)) = \deg(\pi(y^M h_2^M)) \geq M.$$

Moreover, since $M \geq 2$, $\deg(\pi(y^{2M} h_2^{2M})) > \deg(\pi(yh_2))$, and hence

$$\deg(\pi(yh_2 + (x + yh_1 + y^M h_2^M)^2)) = \deg(\pi(y^{2M} h_2^{2M})) > 1$$

which contradicts to $\pi(yh_2 + (x + yh_1 + y^M h_2^M)^2) = z$. Hence $s(z) \notin K$.

Now assume $\deg(s) \cdot \deg(t) > 0$. Also, $\pi(h_2) \neq 0$ since z is not a square. Then $\deg(\pi(yh_2)) \geq 1$ and hence $\deg(\pi(y^M h_2^M)) \geq M \cdot \deg(t(z))$.

By the condition, $\deg(s) \leq N \cdot \deg(t)$, and hence $\deg(\pi(x)) < M \cdot \deg(t)$. Similarly,

$$\begin{aligned} \deg(\pi(yh_1)) &\leq \deg(t) \cdot (1 + \deg_y(h_1)) + \deg(s) \cdot \deg_x(h_1) \\ &\leq \deg(t) + (\deg(s) + \deg(t)) \deg(h_1) \\ &\leq \deg(t) \cdot (1 + (N + 1) \deg(h_1)) \\ &< M \cdot \deg(t), \end{aligned}$$

and hence $\deg(\pi(yh_2 + (x + yh_1 + y^M h_2^M)^2)) = (\deg(\pi(x + yh_1 + y^M h_2^M)))^2 > 1$ which contradicts. Hence to any positive integer n ,

there exists some $\pi_n = (s_n(z), t_n(z))$ where $\deg(s_n) \cdot \deg(t_n) > 0$ and $\deg(s_n) > n \cdot \deg(t_n)$ such that

$$K[\pi'(f), \pi'(g)] = K[z].$$

□

Now we establish the lexicographic order on all monomials of $K[x, y]$ by $x >> y$ and denote the leading monomial of $l(x, y)$ by $v(l)$ for each polynomial $l(x, y) \in K[x, y]$.

Lemma 2.3. *Let $\psi = (f', g')$ be an endomorphism of $K[x, y]$ that maps each coordinate to a generator of some proper retract. If there exists a sequence $(\pi_n = (s_n(z), t_n(z)))$ with $\deg(s_n) \cdot \deg(t_n) > 0$ and $K[\pi_n(f'), \pi_n(g')] = K[z]$ such that $\deg(s_n) > n \cdot \deg(t_n)$, then one of f', g' is of the form $a'y + b'$ where $a', b' \in K$, $a' \neq 0$, or $v(f')$ and $v(g')$ are algebraically dependent with the one of a greater degree being a power of the other one.*

Proof. If neither f' nor g' is of the form $a'y + b'$, then x appears in both f' and g' since a polynomial of outer rank one can not be a generator of a proper retract if the degree is greater than 1. Assume $v(f') = y^a x^b$ and $v(g') = y^c x^d$ where $b \cdot d \neq 0$. Then to any $y^i x^j \in \text{supp}(f')$, if assume $m_n = \deg(s_n) / \deg(t_n)$, we have

$$\frac{\deg(\pi_n(y^i x^j))}{\deg(\pi_n(y^a x^b))} = \frac{i + jm_n}{a + bm_n}$$

where $m_n > n$. Since $y^a x^b$ is the leading term, $j < b$ or $j = b, i < a$, and hence there exists some N_{ij} such that when $n > N_{ij}$, $\deg(\pi_n(y^i x^j)) < \deg(\pi_n(y^a x^b))$. Since there exists finitely many monomials in $\text{supp}(f')$, there exists some N_1 such that to any $n > N_1$ we have $\deg(\pi_n(f')) = \deg(\pi_n(y^a x^b))$. Similar to f' , there exists some N_2 such that to any $n > N_2$ we have $\deg(\pi_n(g')) = \deg(\pi_n(y^c x^d))$. Define $N_0 = \max\{N_1, N_2\}$, and then to any $n > N_0$ we have $\deg(\pi_n(f')) = \deg(\pi_n(y^a x^b))$ and $\deg(\pi_n(g')) = \deg(\pi_n(y^c x^d))$.

Since $K[\pi_n(f'), \pi_n(g')] = K[z]$, by the famous Abhyankar-Moh Theorem ([1], Main Theorem), the greater one of $\deg(\pi_n(f'))$ and $\deg(\pi_n(g'))$ is a multiple of the other one, and hence to any $n > N_0$, we also have

the greater one of $\deg(\pi_n(f'))$ and $\deg(\pi_n(g'))$ is a multiple of the lower one. Without loss of generality, we can assume $b \geq d$, and then

$$\lim_{n \rightarrow \infty} \frac{\deg(\pi_n(f'))}{\deg(\pi_n(g'))} = \frac{b}{d},$$

and if b is not a multiple of d , then there exists some n' such that $\deg(\pi_{n'}(f'))$ is not a multiple of $\deg(\pi_{n'}(g'))$ which contradicts. Hence $d \mid b$, and then

$$\frac{\deg(\pi_n(f'))}{\deg(\pi_n(g'))} = \frac{a \deg(t_n) + b \deg(s_n)}{c \deg(t_n) + d \deg(s_n)} = k + \frac{a - ck}{c + dm_n}$$

where $k = b/d$. If $a - ck \neq 0$, then m_n can be chosen great enough such that $0 < (a - ck)/(c + dm_n) < 1$, and hence $\deg(\pi_n(f'))$ is no longer a multiple of $\deg(\pi_n(g'))$ which contradicts. Hence $a - ck = 0$ which implies that $y^a x^b = (y^c x^d)^k$.

□

Lemma 2.4. *Let $\psi = (f', g')$ be an endomorphism which maps each coordinate to a generator of some proper retract. If there exists some automorphism α such that $\psi \circ \alpha$ is of the form $(x + yh'_1, yh'_2)$, then there exists a sequence $(\pi_n = (s_n(z), t_n(z)))$ with $\deg(s_n) \cdot \deg(t_n) > 0$ and $K[\pi_n(f'), \pi_n(g')] = K[z]$ such that $\deg(s_n) > n \cdot \deg(t_n)$.*

Proof. By Lemma 2.2, this sequence does exist for $\psi \circ \alpha$, and hence it is also a sequence satisfied to ψ .

□

Proof for Main Theorem. Let $\phi = (f, g)$ be an endomorphism which maps each coordinate to a generator of some proper retract. Then it suffices to prove the special case of $f = x + yh_1$ and $g = yh_2$. By Lemma 2.2, there exists a sequence $(\pi_n = (s_n(z), t_n(z)))$ with $\deg(s_n) \cdot \deg(t_n) > 0$ and $K[\pi_n(f), \pi_n(g)] = K[z]$ such that $\deg(s_n) > n \cdot \deg(t_n)$, and by Lemma 2.3 there exists some automorphism α such that $\phi \circ \alpha$ is a reduced form of ϕ . By Lemma 2.4 and Lemma 2.3, the reduction can be done continuously till some component is reduced to a linear form of y . Hence after left and right combine some automorphism, ϕ is transferred to be of the form $\phi' = (y, xh_3)$. Left combine the

automorphism $\beta = (y, x)$, and then $\beta \circ \phi' = (x, yh'_3)$ where $h'_3 \neq 0$. Again by Lemma 2.2 and Lemma 2.3, if h'_3 is not a constant, then x appears in h'_3 and hence $v(yh'_3)$ and x are algebraically dependent. This is impossible, so h'_3 has to be a non-zero constant which implies that $\beta \circ \phi'$ is an automorphism. And then ϕ is an automorphism.

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